Topology Vol. 16, pp. 285–298. Pergamon Press, 1977. Printed in Great Britain

THE COHOMOLOGY OF THE VECTOR FIELDS ON A MANIFOLD

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(*Received* 1 *May* 1977)

§1. INTRODUCTION

THE SMOOTH vector fields on a smooth manifold M form a Lie algebra Vect(M) under the bracket. Gelfand and Fuchs[7–9] have studied the Lie algebra cohomology of Vect(M), which they define by means of a cochain algebra $A(M)$, where $A^k(M)$ is the vector space of continuous **R**-multilinear maps

$$\underbrace{\mathrm{Vect}(M)\times\cdots\times\mathrm{Vect}(M)}_{\leftarrow k\rightarrow}\to\mathbf{C},$$

and the differential d: $A^k(M)\to A^{k+1}(M)$ is defined by the formula

$$\mathrm{d}\alpha(\xi_1,\ldots,\xi_{k+1})=\sum_{i<j}(-1)^{i+j-1}\alpha([\xi_i,\xi_j],\xi_1,\ldots,\hat{\xi}_i,\ldots,\hat{\xi}_j,\ldots,\xi_{k+1}).$$

'Continuous' refers to the usual C^∞ topology on Vect(M). (Actually Gelfand and Fuchs considered the cohomology with real coefficients, but we have found it convenient to change from **R** to **C**.)

In this paper we shall prove that when M is either a compact manifold or the interior of a compact manifold with boundary the cohomology of Vect(M) is the same as that of the space of continuous cross-sections of a certain natural fibre bundle E_M on M associated to its tangent bundle. The fibre of E_M is an open manifold F whose cohomology is that of Vect($\mathbf{R}^n$). The result was conjectured independently by Fuchs and the first author, and has also been proved by Haefliger[11] and Trauber by different methods.

The history of the present proof is roughly as follows. Gelfand and Fuchs in their first highly original papers considered only vector fields with compact supports. They observed that the cochains of the Lie algebra can be regarded as distributions, and they obtained their results by studying the natural filtration of the cochains by the dimension of their supports. Our plan has been to work with general vector fields on non-compact manifolds, so that they can be restricted to open submanifolds. This enables us to prove the theorem by first considering the case of $\mathbf{R}^n$ and then using a patching-together process. We treat $\mathbf{R}^n$ by a contraction argument which we think illuminates the essential homotopy-invariance of the Lie algebra cohomology (cf. Prop. (2.1)). It turns out that the Lie algebra of all vector fields on $\mathbf{R}^n$ has the same cohomology as the jets of vector fields at a single point, i.e., the "formal vector fields" at the origin in $\mathbf{R}^n$. Given the validity of the local result it seemed, especially in view of Anderson's calculation of the cohomology of mapping spaces[1], that a fairly simple patching argument should prove the conjecture. But there were two difficulties: first the technical difficulty that the patching involved surprisingly delicate questions of convergence, and secondly the fundamental difficulty that there seemed to be no map relating the Lie algebra cohomology to that of a space of sections. A solution of these difficulties along the lines presented here was devised, more or less, by the second author in 1974, and has been gradually improved by the authors jointly since then. It is still not very satisfying.

It may be worth commenting on the relationship between our approach and others. Haefliger's framework is Sullivan's explicit algebraic model of the rational homotopy category in terms of minimal differential graded algebras. Roughly one can say that he constructs on the one hand a minimal model for the Lie algebra cochains and on the other hand a minimal model for the homotopy type of the space of sections, and proves that the two models are actually isomorphic. An advantage of his approach is that it incorporates an explicit calculation of the cohomology, at least for manifolds whose Pontrjagin classes vanish. Our method, on the other hand, is more self-contained, and relates the Gelfand–Fuchs cochains comparatively directly to the cochain algebra of the space of sections. It has the advantage of incorporating a proof that the cohomology is finite dimensional, a result due to Gelfand and Fuchs in the case of vector fields with compact supports.

Trauber's method is to use the Gelfand–Fuchs filtration. Its relation to Haefliger's approach is roughly that of Quillen's[13] to Sullivan's[16] treatment of the rational homotopy category.

Concerning the general nature of the problem one can say the following. If it were possible to associate a classifying space $B\mathfrak{g}$ naturally to a Lie algebra $\mathfrak{g}$ so that $H^*(\mathfrak{g}) = H^*(B\mathfrak{g})$ then the theorem would be rather easy. For the points of the principal bundle P of a Riemannian manifold M define exponential maps $\mathbf{R}^n \to M$ which induce homomorphisms of Lie algebras $\mathrm{Vect}(M) \to V_n$, where V_n is the formal vector fields at O in $\mathbf{R}^n$. Applying the functor B would give a map $P \to \mathrm{Map}(B\mathrm{Vect}(M); BV_n)$ equivariant with respect to the action of O_n on P and BV_n. By adjunction we should have $B\mathrm{Vect}(M) \to \mathrm{Map}_{O_n}(P; BV_n)$. But $\mathrm{Map}_{O_n}(P; BV_n)$ is the space of sections of a bundle on M with fibre BV_n, associated to P. One could hope that the last map was a cohomology equivalence.

But unfortunately no such classifying space functor $\mathfrak{g} \mapsto B\mathfrak{g}$ can possibly exist. If it did there would be a natural integral lattice in the cohomology of a Lie algebra. In fact $B\mathfrak{g}$ exists as a real homotopy type, but then it is its "continuous" cohomology that is relevant for us. To make clear the significance of this we should point out that our result is closely related to the fact that the cohomology of the Lie algebra $\mathfrak{g}$ of a reductive Lie group G is (with complex coefficients) that of the *complexification* $G_{\mathbf{C}}$ of G as a space, by the maps

$$\begin{aligned}(\text{cochains of } \mathfrak{g}) &= (\text{invariant forms on } G)\\ &= (\text{holomorphic invariant forms on } G_{\mathbf{C}}) \subset (\text{forms on } G_{\mathbf{C}}).\end{aligned}$$

On the other hand if one associates a simplicial set abstractly to the cochains of $\mathfrak{g}$ following Sullivan's general prescription[16] one gets the quotient by G of the usual singular complex of G. The homotopy type of this is $(EG_\delta \times G)/G_\delta$, where G_δ is G with the discrete topology and EG_δ is its universal space. This is the homotopical fibre of the map $BG^\delta \to BG$, and there seems to be no obvious relation between it and $G_{\mathbf{C}}$.

The plan of the paper is as follows. In §2 we study $A(\mathbf{R}^n)$, proving in particular that its cohomology is finite dimensional. In §3 we construct the manifold F whose cohomology is that of $A(\mathbf{R}^n)$. In §4 we construct our fundamental map. In §5 we describe without proof how to construct a cochain algebra representing a mapping space or a space of sections. The result here is essentially due to Anderson[1], but we give a self-contained treatment which we hope is of some interest in its own right. The main theorem is stated in §6, and proved modulo a convergence lemma whose proof is postponed to §8. The result of §5 is proved in §7.

§2. THE CASE OF EUCLIDEAN SPACE

The radial contractions $T_t: \mathbf{R}^n \to \mathbf{R}^n$ defined by $T_t(x) = tx$ (for $0 < t \le 1$) induce endomorphisms of $A(\mathbf{R}^n)$ which induce the identity on cohomology. For inner automorphisms of a Lie algebra induce the identity on its cohomology, and as the one-parameter group of diffeomorphisms $\{T_t\}$ is generated by a vector field ρ the derivative with respect to t of the action of T_t on $A(\mathbf{R}^n)$ comes from the inner automorphism of $\mathrm{Vect}(\mathbf{R}^n)$ defined by ρ. (This will be made more explicit in the proof of (2.2) below.)

Now for any manifold M an element of $A^q(M)$, being a multilinear function on smooth vector fields on M, can be thought of as a distribution with compact support on the product of q copies of M, with values in some appropriate vector bundle. The contractions T_t replace the cocycles of $A(\mathbf{R}^n)$ by cohomologous ones with smaller support.

It follows from this that,

PROPOSITION 2.1. *If U is a star-shaped open set of $\mathbf{R}^n$ then $A(U) \to A(\mathbf{R}^n)$ is a cohomology equivalence.*

But it is natural to ask whether one can pass to the limit as $t \to 0$ and obtain cocycles whose support is a single point. We shall prove that that is so.

PROPOSITION 2.2. *Let $A_0(\mathbf{R}^n)$ be the subalgebra of $A(\mathbf{R}^n)$ consisting of cochains whose support is the origin. Then the inclusion $A_0(\mathbf{R}^n)\to A(\mathbf{R}^n)$ is a cohomology equivalence.*

Remark. Associating to a vector field on $\mathbf{R}^n$ its Taylor expansion at the origin is a continuous surjection of Lie algebras from $\mathrm{Vect}(\mathbf{R}^n)$ to the algebra $\hat{V}_n$ of formal vector fields at the origin in $\mathbf{R}^n$. The subalgebra $A_0(\mathbf{R}^n)$ is exactly the continuous cochain algebra of $\hat{V}_n$. It has a purely algebraic description, as we shall see in §3.

Proof of 2.2. We shall actually prove considerably more, namely that the cohomology class of a cochain α is determined by its values on a certain finite dimensional space of polynomial vector fields.

Let us write V for $\mathbf{R}^n$. A vector field on V is an element of $\mathscr{E}(V)\otimes V$, where $\mathscr{E}(V)$ is the smooth real-valued functions on V. So an element of $A^1(V)$ is a continuous linear map $\mathscr{E}(V)\to V^*$, and more generally an element of $A^q(V)$ is a continuous multilinear map

$$\alpha\colon \underbrace{\mathscr{E}(V)\times\cdots\times\mathscr{E}(V)}_{\leftarrow q\rightarrow}\to(V^*)^{\otimes q}$$

which is alternating in the obvious sense. (Note that as we are using complex coefficients the dual V^* means $\mathrm{Hom}_{\mathbf{R}}(V;\mathbf{C})$.) In this notation $\mathrm{d}\alpha$ is the multilinear map obtained by making alternating the map

$$(f_1,\ldots,f_{q+1})\to\sum_i \alpha\Bigl(\underbrace{\frac{\partial f_1}{\partial x_i}\cdot f_2, f_3,\ldots,f_{q+1}}_{\leftarrow q\rightarrow}\Bigr)\otimes e_i^*,$$

where $\{e_i^*\}$ is the standard basis of V^*.

Let P_q be the finite dimensional subspace of $\mathscr{E}(V)\otimes\cdots\otimes\mathscr{E}(V)$ spanned by all elements $f_1\otimes\cdots\otimes f_q$, where $f_1,\ldots,f_q$ are polynomials such that $\Sigma\,\mathrm{degree}\,(f_i)\le q$. If $A_P{}^q(V)$ denotes the alternating linear maps $P_q\to(V^*)^{\otimes q}$ the preceding formula for d defines $\mathrm{d}\colon A_P{}^q(V)\to A_P{}^{q+1}(V)$, making $A_P(V)$ a quotient cochain complex of $A(V)$. Let $B(V)$ be the kernel of $A(V)\to A_P(V)$.

LEMMA 2.3. *$B(V)$ is acyclic, and hence $A(V)\to A_P(V)$ is a cohomology equivalence.*

Remark. Because $A_P{}^q(V)$ is finite dimensional this implies that the cohomology groups of $A(V)$ are finite dimensional.

Proof. In the above notation the contraction $T_t\colon V\to V$ acts on $A(V)$ by $(T_t\alpha)(f_1,\ldots,f_q)=t^{-q}\alpha(T_t^*f_1,\ldots,T_t^*f_q)$. If $\alpha\in B(V)$ then $T_t\alpha\to0$ as $t\to0$, for if f vanishes to order k at the origin then T_t^*f is $\mathrm{O}(t^k)$ as $t\to0$, and any function f is the sum of a polynomial of degree $k-1$ and a function which vanishes to order k. (In fact $B(V)$ consists precisely of all $\alpha\in A(V)$ such that $T_t\alpha\to0$ as $t\to0$.)

Now because the group of diffeomorphisms $\{T_t\}$ is generated by a vector field ρ on V the derivative $T_t^{-1}\cdot t(\mathrm{d}/\mathrm{d}t)T_t$ is the Lie derivative $\theta(\rho)\colon A(V)\to A(V)$ along ρ. (One has $t(\mathrm{d}/\mathrm{d}t)$ rather than $\mathrm{d}/\mathrm{d}t$ because the group T_t is written multiplicatively rather than additively.) But for any Lie algebra

$$\theta(\rho)=i(\rho)\,\mathrm{d}+\mathrm{d}i(\rho),$$

where $i(\rho)\colon A(V)\to A(V)$ is the inner product with ρ. So

$$\frac{\mathrm{d}}{\mathrm{d}t}\,T_t\alpha=H_t\,\mathrm{d}\alpha+\mathrm{d}H_t\alpha,$$

where $H_t=t^{-1}T_ti(\rho)=i(\rho)t^{-1}T_t$. Then if $T_t\alpha$ is $\mathrm{O}(t)$ as $t\to0$ we have

$$\alpha=T_1\alpha-T_0\alpha=K\,\mathrm{d}\alpha+\mathrm{d}\,K\alpha,$$

where $K\alpha=i(\rho)\int_0^1 t^{-1}T_t\alpha\cdot\mathrm{d}t$. Thus K is a contracting homotopy for $B(V)$.

Conclusion of proof of (2.1). $A_P(V)$ is a quotient of $A_0(V)$ as well as of $A(V)$. The kernel of $A_0(V)\to A_P(V)$ is $B_0(V)=B(V)\cap A_0(V)$. But the contracting homotopy K preserves $B_0(V)$, so $A_0(V)\to A_P(V)$ is a cohomology equivalence as well as $A(V)\to A_P(V)$.

Complement. The preceding argument can clearly be applied not only to $\mathbf{R}^n$ but to a finite disjoint union of copies of $\mathbf{R}^n$, and then will prove that the inclusion $A_0(\amalg^k \mathbf{R}^n) \to A(\amalg^k \mathbf{R}^n)$ is a cohomology equivalence, where $\amalg^k \mathbf{R}^n$ means $\underbrace{\mathbf{R}^n \amalg \dots \amalg \mathbf{R}^n}_{\leftarrow k \rightarrow}$ and $A_0(\amalg^k \mathbf{R}^n)$ means the cochains whose supports involve only the origins in the various $\mathbf{R}^n$'s. This remark allows us to avoid using a Künneth theorem for topological vector spaces, for $A_0(\amalg^k \mathbf{R}^n)$ is just the algebraic tensor product $A_0(\mathbf{R}^n)^{\otimes k}$, and its cohomology is $H^*(A_0(\mathbf{R}^n))^{\otimes k}$ by the ordinary Künneth theorem.

§3. THE MANIFOLD F

Our next task is to construct a smooth manifold F whose cohomology is the same as that of $A(\mathbf{R}^n)$. More precisely we shall construct F and a cochain algebra A and maps of cochain algebras

$$A(\mathbf{R}^n) \leftarrow A \rightarrow \Omega(F)$$

which induce isomorphisms of cohomology. (Here $\Omega(F)$ denotes the differential forms on F). But we shall need to construct F so that the orthogonal group $G = O_n$ acts on it compatibly with its natural action on $A(\mathbf{R}^n)$. To explain this entails digressing to outline the theory of group actions on cochain algebras developed by Cartan in [3].

Definition (3.1). To say that a Lie group G acts on a topological cochain algebra A means that

(i) G acts smoothly on A by automorphisms, in the obvious sense, and

(ii) for each ξ in the Lie algebra $\mathfrak{g}$ of G there is given an antiderivation $i(\xi)\colon A \to A$ of degree -1 such that $\mathrm{d}\, i(\xi) + i(\xi)\,\mathrm{d} = \theta(\xi)$, where $\theta(\xi)$ is the derivation of degree 0 got by differentiating the G-action along the tangent vector ξ at the identity, and

(iii) $i(g \cdot \xi) = g \cdot i(\xi) \cdot g^{-1}$, where $g \cdot \xi$ denotes the adjoint action of $G \in G$ on $\xi \in \mathfrak{g}$.

A G-homomorphism of algebras with such G-action is one which commutes with the derivations $i(\xi)$ for all $\xi \in \mathfrak{g}$, as well as with the elements of G.

If G acts on a manifold X it acts on both $\Omega(X)$ and $A(X)$ in the sense of (3.1).

The G-cochain algebras we shall encounter resemble the algebra of differential forms on a smooth principal G-bundle P. In that case the G-action on P identifies $\mathfrak{g}$ with a subspace of the tangent space to P at each point, and one can choose an embedding $\omega\colon \mathfrak{g}^* \to \Omega^1(P)$ such that $i(\xi)\omega(\alpha) = \langle \alpha, \xi \rangle$ for $\xi \in \mathfrak{g}$, $\alpha \in \mathfrak{g}^*$. One can even make ω G-equivariant. To do so is exactly to choose a connection in P in the classical sense: one often thinks of ω as a G-invariant element of $\Omega^1(P; \mathfrak{g})$. It is explained in [3] that ω extends canonically to a G-homomorphism of cochain algebras $W \to \Omega(P)$, where W is the *Weil algebra* of G.

Definition (3.2). The Weil algebra W of G is $\Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$ as algebra, where the exterior algebra $\Lambda\mathfrak{g}^*$ is generated by 1-forms $\alpha \in \mathfrak{g}^*$, and the symmetric algebra $S\mathfrak{g}^*$ by 2-forms Ω_α for $\alpha \in \mathfrak{g}^*$.

Its differential is defined by $\mathrm{d}\alpha = \mathrm{d}_1\alpha + \Omega_\alpha$, where $\mathrm{d}_1\alpha \in \Lambda^2\mathfrak{g}^*$ is the differential of α in the cochain complex of the Lie algebra $\mathfrak{g}$.

Its G-action is defined by making $i(\xi)$ (for $\xi \in \mathfrak{g}$) act as the obvious anti-derivation on $\Lambda\mathfrak{g}^*$, and trivially on $S\mathfrak{g}^*$.

The classical theory makes the following definition reasonable.

Definition (3.3). A *connection* in a G-cochain algebra A is a G-homomorphism $W \to A$, where W is the Weil algebra of G.

In the case of the forms on a principal bundle P choosing a connection permits one to identify $\Omega(P)$ as an algebra with $\Lambda\mathfrak{g}^* \otimes \Omega_{\mathrm{horiz}}(P)$, where $\Omega_{\mathrm{horiz}}(P)$ is the horizontal forms on P, i.e. those annihilated by $i(\xi)$ for all $\xi \in \mathfrak{g}$. (But notice that $\Omega_{\mathrm{horiz}}(P)$ is not closed under d.) In other words the connection $W \to \Omega(P)$ induces an isomorphism of algebras $W \otimes_{S\mathfrak{g}^*} \Omega_{\mathrm{horiz}}(P) \to \Omega(P)$. This motivates

Definition (3.4). A G-cochain algebra A with a connection $W \to A$ is *standard* if $W \otimes_{S\mathfrak{g}^*} A_{\mathrm{horiz}} \to A$ is an isomorphism.

If A is a G-cochain algebra its *basic* subalgebra A_{basic} consists of the G-invariant horizontal elements. Unlike A_{horiz} it is closed under d. If $A = \Omega(P)$, where P is a principal G-bundle, then A_{basic} can be identified with $\Omega(P/G)$. In the case of the Weil algebra W_{basic} is the invariant part of $S\mathfrak{g}^*$, which we shall call C. When $G = GL_n(\mathbf{R})$ it is well-known that C is the polynomial algebra on a sequence of elements c_i of dimension $2i(1 \le i \le n)$, called the universal Chern classes. The differential is zero.

A connection $W \to A$ in a G-cochain algebra induces a G-homomorphism of cochain algebras $W \otimes_C A_{\text{basic}} \to A$. The only technical result we need here is

PROPOSITION (3.5). *If G is connected and reductive, and A is a standard G-cochain algebra which as G-module is the sum of isotypical pieces corresponding to finite dimensional irreducible G-modules then $W \otimes_C A_{\text{basic}} \to A$ induces an isomorphism of cohomology. Furthermore a G-homomorphism $A \to A'$ of such algebras induces an isomorphism of cohomology if $A_{\text{basic}} \to A'_{\text{basic}}$ does.*

We shall leave the proof to the end of this section. One situation to which the proposition applies is when G is compact and $A = \Omega(P)$ is the forms on a principal G-bundle. Then it is an algorithm for calculating $H^*(P)$ from $\Omega(P/G)$ and the characteristic classes $C \to \Omega(P/G)$. Cartan[3] proved a slightly different result, which is much more useful for calculation, namely that there is a cohomology equivalence $E \otimes_C A_{\text{basic}} \to A$, where $E = \Lambda \otimes C$ as algebra, and Λ is the exterior algebra on elements $e_1, \ldots, e_n$ with $de_i = c_i$. Unfortunately that is of no use here as E is not a G-algebra.

The algebra $A_0(\mathbf{R}^n)$ of continuous cochains on the Lie algebra V_n of formal vector fields at the origin in $\mathbf{R}^n$ is a G-cochain algebra, where $G = GL_n(\mathbf{R})$, because the Lie algebra $\mathfrak{g}$ of G is a subalgebra of V_n. It has a natural connection, which makes it a standard G-algebra, because there is a G-stable splitting $V_n = \mathfrak{g} \oplus W$. In fact one can decompose V_n as $\prod_{k \ge -1} V_n^k$, where V_n^k is the part which is homogeneous of degree k under radial expansion, and $\mathfrak{g} = V_n^0$ and $W = \prod_{k \ne 0} V_n^k$. Gelfand and Fuchs[9] calculated $A_0(\mathbf{R}^n)_{\text{basic}}$, and proved it is generated by the characteristic classes, more precisely,

PROPOSITION (3.6) (Gelfand–Fuchs). *The connection $W \to A_0(\mathbf{R}^n)$ induces an isomorphism $\bar{C} \to A_0(\mathbf{R}^n)_{\text{basic}}$, where $\bar{C}$ is the quotient of the polynomial algebra $C = W_{\text{basic}}$ on $c_1, \ldots, c_n$ by the ideal of all elements of dimension greater than $2n$.*

We shall not prove this here (cf. [10]), but shall make some remarks about the proof. V_n has the topological basis $\{x_{j_0} \ldots x_{j_k}(\partial/\partial x_i)\}$, where $(i, j_0, \ldots, j_k)$ runs through the $(k+2)$-tuples such that $j_0 \le j_1 \le \cdots \le j_k$. Let $\{\theta^i_{j_0 \ldots j_k}\}$ be the dual basis of V_n^*. Then $A = A_0(\mathbf{R}^n)$ is the exterior algebra on the $\theta^i_{j_0 \ldots j_k}$, and A_{horiz} is generated by those with $k \ne 0$. The connection forms are the θ_j^i, and from the relation $d\theta_j^i = \Sigma \theta_k^i \wedge \theta_j^k + \Sigma \theta^k \wedge \theta^i_{jk}$ it follows that the curvature forms (the images of the generators of $S\mathfrak{g}^* \subset W$) are given by $\Omega_j^i = \Sigma \theta^k \wedge \theta^i_{jk}$. From invariant theory (cf. [2]) one knows that A_{basic} is generated by completely contracted expressions such as

$$\Omega_1 = \Sigma \Omega_i^i = \Sigma \theta^i \wedge \theta^i_{ij}$$

$$\Omega_2 = \Sigma \Omega_j^i \wedge \Omega_i^j = \Sigma \theta^i \wedge \theta^j \wedge \theta^i \wedge \theta^k_{il} \wedge \theta^l_{jk}$$

$$\Omega_n = \Sigma \Omega^{i_1}_{i_2} \wedge \cdots \wedge \Omega^{i_n}_{i_1} = \Sigma \theta^{j_1} \wedge \cdots \wedge \theta^{j_n} \wedge \theta^{i_1}_{i_2 j_1} \wedge \cdots \wedge \theta^{i_n}_{i_1 j_n}.$$

Because $\theta^i_{j_0 \ldots j_k}$ is symmetric in its lower indices it is easy to see that it cannot occur in any invariant expression if $k > 1$, and that any invariant is a polynomial in $\Omega_1, \ldots, \Omega_n$. These are essentially the Chern classes. The difficult step is to see that there are no relations among them except the obvious vanishing of expressions of dimension greater than $2n$.

In view of (3.5) if we could find a manifold B with a cohomology equivalence $\bar{C} \to \Omega(B)$ we could construct a bundle on B whose cohomology would be that of $A_0(\mathbf{R}^n)$. But is enough to find B and a cochain algebra M with equivalences $\bar{C} \leftarrow M \to \Omega(B)$. In fact *any* manifold B whose cohomology ring is $\bar{C}$ will do, for Sullivan's theory[16] shows that there is a unique minimal cochain algebra M (in his sense) whose cohomology is $\bar{C}$, and this M will automatically map to $\bar{C}$ and $\Omega(B)$. In other words the rational homotopy type of B is completely determined by its cohomology ring. There is in any case a natural candidate for B. The complex Grassmannian $G_{n,2n}$ of n-dimensional subspaces of $\mathbf{C}^{2n}$ has its cohomology ring generated by the Chern classes $c_1,\ldots,c_n$, and there are no relations between them up to dimension $2n$. It also has a natural cell decomposition given by its Schubert cells, which are complex manifolds and hence even-dimensional. The union B_0 of all the Schubert cells of $G_{n,2n}$ of dimension $\leq 2n$ has the desired cohomology ring $\bar{C}$. But as B_0 is an algebraic variety with singularities we take B to be an open neighborhood of B_0 in $G_{n,2n}$ to get a smooth manifold.

One further point needs to be mentioned. One cannot apply (3.5) to the forms on a principal G-bundle when G is not compact, as the condition about finite-dimensional irreducible G-modules does not hold. But *with complex coefficients* (which we are using) the Weil algebras of $GL_n(\mathbf{R})$ and U_n are identical, for invariant forms on either can be identified with holomorphic invariant forms on $GL_n(\mathbf{C})$. So we define F to be the natural principal U_n-bundle on the partial Grassmannian B, and then have cohomology equivalences

$$A_0(\mathbf{R}^n) \leftarrow W \otimes_C \bar{C} \leftarrow W \otimes_C M \rightarrow W \otimes_C \Omega(B) \rightarrow \Omega(F),$$

where W is the common Weil algebra of $GL_n(\mathbf{R})$ and U_n. The basic subalgebra of W is C with respect to any of the three groups $GL_n(\mathbf{R})$, $GL_n^+(\mathbf{R})$, U_n. ($GL_n^+(\mathbf{R})$ is the identity component of $GL_n(\mathbf{R})$.) And the basic subalgebra of $A_0(\mathbf{R}^n)$ is $\bar{C}$ for both $GL_n(\mathbf{R})$ and $GL_n^+(\mathbf{R})$. So the two maps on the right are U_n-equivalences by (3.5), and the two on the left are $GL_n^+(\mathbf{R})$-equivalences by (3.5), and hence $GL_n(\mathbf{R})$-equivalences. Thus all the maps are compatible with $O_n = GL_n(\mathbf{R}) \cap U_n$, and, taking $A = W \otimes_C M$ we have the desired O_n-equivalences

$$A_0(\mathbf{R}^n) \leftarrow A \rightarrow \Omega(F).$$

We conclude this section by giving the proof of Proposition (3.5).

We are given that $A \cong \Lambda\mathfrak{g}^* \otimes A_{\text{horiz}}$. Then A has a filtration $A = A_0 \supset A_1 \supset A_2 \supset \ldots$, where $A_p = \bigoplus_{q \geq p} \Lambda\mathfrak{g}^* \otimes A^q_{\text{horiz}}$, which is compatible with d. It is easy to check that the differential induced in the associated graded algebra gr A makes gr A into $C^*(\mathfrak{g}; A_{\text{horiz}})$, the cochain algebra of the Lie algebra $\mathfrak{g}$ acting on the graded $\mathfrak{g}$-module A_{horiz}. Thus there is a spectral sequence

$$H^*(\mathfrak{g}; A_{\text{horiz}}) \Rightarrow H^*(A).$$

On the other hand the horizontal part of $W \otimes_C A_{\text{basic}}$ is $S\mathfrak{g}^* \otimes_C A_{\text{basic}}$, so there is a similar spectral sequence

$$H^*(\mathfrak{g}; S\mathfrak{g}^* \otimes_C A_{\text{basic}}) \Rightarrow H^*(W \otimes_C A_{\text{basic}}),$$

and a map from the second to the first. But for a reductive algebra $\mathfrak{g}$ and a finite dimensional simple $\mathfrak{g}$-module M the cohomology. $H^*(\mathfrak{g}; M)$ is zero unless $\mathfrak{g}$ acts trivially on M. As the $\mathfrak{g}$-invariant part of $S\mathfrak{g}^* \otimes_C A_{\text{basic}}$ is $C \otimes_C A_{\text{basic}} = A_{\text{basic}}$, which coincides with the invariant part of A_{horiz}, the two spectral sequences have the same E_1-term $H^*(\mathfrak{g}; \mathbf{R}) \otimes A_{\text{basic}}$, and so $H^*(W \otimes_C A_{\text{basic}}) \xrightarrow{\cong} H^*(A)$. The last assertion of (3.5) follows from the fact that the E_2-term of the spectral sequence is $H^*(\mathfrak{g}; \mathbf{R}) \otimes H^*(A_{\text{basic}})$.

§4. THE FUNDAMENTAL MAP

Suppose that M and N are smooth manifolds of the same dimension n, and let P be a smooth family of immersions of M in N. That is, there is a smooth map $f: P \times M \rightarrow N$ such that $f_p = f|p \times M$ is an immersion for each p in P. The purpose of this section is to show that f induces a homomorphism of cochain algebras $\hat{f}: A(M) \rightarrow \Omega(P; A(N))$. The object on the right is the bigraded algebra of differential forms on P with coefficients in $A(N)$. Its differential is the sum of the de Rham differential d_P in $\Omega(P)$ and the Lie algebra differential d_A in $A(N)$.

Before constructing $\hat{f}$ we shall make some general remarks about it. First, if one restricts it to a point $p \in P$ the result is the map $A(M) \rightarrow A(N)$ induced by f_p. If one composes it with the augmentation $A(N) \rightarrow \mathbf{C}$ one obtains a map $A(M) \rightarrow \Omega(P)$ which is also familar. For recall that for any foliation of $P \times M$ of codimension n which is transversal to the fibres of the projection $P \times M \rightarrow P$ there is a "characteristic class" homomorphism $A(M) \rightarrow \Omega(P)$. Our map is associated to the foliation of $P \times M$ by the fibres of $f: P \times M \rightarrow N$.

If G is a Lie group the product $G \times G \to G$ induces a map $\Omega(G) \to \Omega(G \times G) = \Omega(G; \Omega(G))$. Restricting this to forms invariant under the right action of G it becomes $A(\mathfrak{g}) \to \Omega(G; A(\mathfrak{g}))$, for the right-invariant forms on G are the cochains of its Lie algebra $\mathfrak{g}$. The map we shall construct is an analogue of this. For $f: P \times M \to N$ induces a map $P \times \mathrm{Imm}(N; N) \to \mathrm{Imm}(M; N)$ which is equivariant with respect to the monoid $\mathrm{Imm}(N; N)$. ("Imm" denotes a space of immersions.) But $A(M)$ and $A(N)$ can be thought of as in some sense the differential forms on $\mathrm{Imm}(M; N)$ and $\mathrm{Imm}(N; N)$ which are left-invariant under $\mathrm{Imm}(N; N)$.

The construction is as follows. The tangent space to $\mathrm{Imm}(M; N)$ at any point can be identified with the vector fields on M. For if $f_t: M \to N$ is a one-parameter family of immersions then $m \to \{Df_t(m)^{-1}(\mathrm{d}/\mathrm{d}t)f_t(m)\}|_{t=0}$ is a vector field on M. Accordingly, given a family $f_p: M \to N$ parametrized by $p \in P$ one can associate to each tangent vector to P a vector field on M. I.e. one has a map $\omega_p: P \to \mathrm{Vect}(M)$ for each $p \in P$. On the other hand f_p induces $f_p^*: \mathrm{Vect}(N) \to \mathrm{Vect}(M)$. Now elements of $\Omega(P; A(N))$ can be thought of as functions which to each point p of P associate a multi-linear alternating form on $T_pP \oplus \mathrm{Vect}(N)$. Starting with $\alpha \in A(M)$ we get such a function by composing α with the family of maps $(-\omega_p) \oplus f_p^*: T_pP \oplus \mathrm{Vect}(N) \to \mathrm{Vect}(M)$. This defines a homomorphism of algebras $\hat{f}: A(M) \to \Omega(P; A(N))$. We have to prove that it is compatible with the differentials. It is enough to do so for elements of $A^1(M)$, for their products are dense in $A(M)$, and the differentials are antiderivations.

We shall think of the maps ω_p as constituting a 1-form ω on P with values in $\mathrm{Vect}(M)$, and of the f_p^* as a 0-form with values in $\mathrm{Hom}(A(M); A(N))$. Then the definition of ω can be re-expressed as $d_Pf = -f \cdot \theta(\omega)$, where $\theta(\omega)$ is thought of as a 1-form with values in $\mathrm{End}(A(M))$. Applying d_P to this definition gives the "Maurer–Cartan" relation $\mathrm{d}_P\theta(\omega) = -\theta(\omega) \wedge \theta(\omega)$, because f is injective at each point. Alternatively expressed, $\mathrm{d}_P\omega = -\frac{1}{2}[\omega, \omega]$.

For $\alpha \in A^1(M)$ we have defined $\hat{f}\alpha = f \cdot \alpha - \alpha(\omega)$. So

$$\begin{aligned}
\mathrm{d}\hat{f}\alpha &= \mathrm{d}_Pf \cdot \alpha + \mathrm{d}_Af\alpha - \mathrm{d}_P\alpha(\omega) \quad (\text{for } \mathrm{d}_A = 0 \text{ on } A^0(N)) \\
&= -f \cdot \theta(\omega) \cdot \alpha + f\mathrm{d}_A\alpha + \frac{1}{2}\alpha([\omega, \omega]) \\
&= -f \cdot i(\omega)\mathrm{d}_A\alpha + f\mathrm{d}_A\alpha + \frac{1}{2}\mathrm{d}_A\alpha(\omega, \omega) \\
&= f \cdot (1 - i(\omega) + \frac{1}{2}i(\omega)i(\omega)) \cdot \mathrm{d}A\alpha \\
&= \hat{f}\mathrm{d}_A\alpha,
\end{aligned}$$

because for $\beta \in A^2(M)$ one has $\hat{f}\beta = f \cdot (1 - i(\omega) + \frac{1}{2}i(\omega)i(\omega)) \cdot \beta$. (The general formula, of course, is $\hat{f} = f \cdot \exp(-i(\omega))$.)

To conclude this section we need one more remark. If a Lie group G acts on P and N so that $f: P \times M \to N$ is equivariant then $\Omega(P; A(N))$ is a G-cochain-algebra in the sense of §3 and $\hat{f}: A(M) \to \Omega(P; A(N))$ has its image contained in $\Omega(P; A(N))_{\text{basic}}$. Indeed the image consists of G-invariant elements by the naturality of the construction, and of horizontal elements because $\hat{f}$ is induced by maps $(-\omega_p) \oplus f_p^*: T_pP \otimes \mathrm{Vect}(N) \to \mathrm{Vect}(M)$ which vanish on the image of $\mathfrak{g}$.

§5. COCHAINS FOR MAPPING SPACES AND SPACES OF SECTIONS

In this section we shall describe how to construct a cohain algebra from which the cohomology of a mapping space, or more generally the space of cross-sections of a fibre bundle, can be calculated.

Let us begin for simplicity with the space of maps $X \to Y$, where X is a space which has an open covering $\mathfrak{u} = \{U_\alpha\}_{\alpha \in S}$ by contractible sets all of whose finite intersections are also contractible or empty. We shall call such a $\mathfrak{u}$ a *contractible covering*. We write $U_\alpha = U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$ when $\sigma = \{\alpha_0, \ldots, \alpha_q\}$ is a subset of S, and also $\sigma \leq \tau$ to denote $\sigma \supset \tau$, so that $\sigma \leq \tau \Rightarrow U_\sigma \subset U_\tau$.

The covering $\mathfrak{u}$ has a *nerve*, which is the simplicial set Σ whose set of p-simplexes Σ_p is the set of chains $\sigma_0 \leq \cdots \leq \sigma_p$ of finite subsets of S such that U_{σ_0} is non-empty. (Of course, this is the barycentric subdivision of what is usually called the nerve.)

We shall prove that in appropriate circumstances the cohomology of the mapping space Y^X is the total cohomology of a double complex

$$C(Y^{\Sigma}) = \{C(Y^{\Sigma_0}) \leftarrow C(Y^{\Sigma_1}) \leftarrow C(Y^{\Sigma_2}) \leftarrow \cdots\}$$

where Y^{Σ_p} denotes the product of a collection of copies of Y indexed by the set Σ_p of p-complexes of the nerve of $\mathfrak{U}$. The horizontal differential in $C(Y^{\Sigma})$ comes from the boundary operators in the nerve Σ. C can denote any reasonable kind of cochains.

To make this more precise it is best to divide the construction into two steps. Using the simplicial set $\{\Sigma_p\}$ we constructed a *covariant* functor $[p] \mapsto Y^{\Sigma_p}$ from the category of finite ordered sets to spaces. Such a functor is called a *cosimplicial space*. To any cosimplicial space $[p] \mapsto Z_p$ is associated a space $|Z|$, called its *realization*. This is defined as the subspace of the product $\prod_{p \geq 0} Z_p^{\Delta_p}$ consisting of all sequences $\alpha_p \colon \Delta^p \to Z_p$ such that the diagram

$$\begin{array}{ccc} \Delta^p & \xrightarrow{\alpha_p} & Z_p \\ {\scriptstyle \theta_*}\downarrow & & \downarrow{\scriptstyle \theta_*} \\ \Delta^q & \xrightarrow{\alpha_q} & Z_q \end{array}$$

commutes for all simplicial operations $\theta \colon [p] \to [q]$. (Here Δ^p denotes the standard p-simplex.) With this definition we have the following tautology.

PROPOSITION (5.1). *The realization of the cosimplicial space $[p] \mapsto Y^{\Sigma_p}$ is exactly the space of maps from the realization of the simplicial set $[p] \mapsto \Sigma_p$ to Y. That is $|Y^{\Sigma}| = Y^{|\Sigma|}$.*

For a contractible covering $\mathfrak{U}$ the nerve $|\Sigma|$ is homotopy equivalent to X (cf. [14, (4.1)] and [15, (A.1)]), so then $Y^X \simeq |Y^{\Sigma}|$.

If Z is a *simplicial* space, i.e. a contravariant functor $[p] \mapsto Z_p$ it is well known that the cohomology of its realization is that of the double complex $\{C^q(Z_p)\}$. What we need here is an analogous statement for cosimplical spaces, but it is very much more delicate, and not true without restrictive hypotheses.

The first thing to notice is that taking the cochains on a cosimplical space Z gives one a simplicial cochain complex $[p] \mapsto C(Z_p)$, i.e. an array of groups

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \uparrow & & \\ C_0^1 & \leftleftarrows & C_1^1 & \Lleftarrow & C_2^1 & \Lleftarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ C_1^0 & \leftleftarrows & C_1^0 & \Lleftarrow & C_1^0 & \Lleftarrow & \cdots, \end{array}$$

where $C_p^q = C^q(Z_p)$. In the total complex C_p^q has degree $q - p$, so in general there will be terms with both positive and negative total degree, whereas the cohomology of $|Z|$ is by definition all of positive degree. In any case we must be quite precise about the definition of the total complex.

Definition (5.2). The *realization* $|C|$ of a simplicial cochain complex $C = \{C_0 \leftleftarrows C_1 \Lleftarrow \cdots\}$ is the total cochain complex of the normalized bicomplex $\{C_0 \leftarrow \bar{C}_1 \leftarrow \bar{C}_2 \leftarrow \cdots\}$, whose vertical differentials are those of the cochain complexes C_p and whose horizontal differentials are the alternating sums of the simplicial operations $\partial_i \colon C_p \to C_{p-1}$.

Normalized means that C_p has been replaced by $C_p = C_p/C_p^{\text{degen}}$, where C_p^{degen} is the sum of the images of the degeneracy operations $s_i \colon C_{p-1} \to C_p$. The *total complex* of the bicomplex has by definition in degree r the direct *sum* (not the product) of the $\bar{C}_p^q$ for $q - p = r$.

We want a theorem asserting that the realization $|C(Z)|$ of the cochains on a cosimplicial space Z has the same cohomology as the realization $|Z|$. Applying it to $Z = Y^{\Sigma}$ will give the desired model for the cochains of the mapping space $Y^{|\Sigma|}$.

Before stating the result we should mention multiplicative properties. We should like to consider not cochain complexes but cochain algebras, for in the light of the theories of

Quillen[13] and Sullivan[16] commutative cochain algebras over the real numbers or the rationals can be thought of as real or rational homotopy types. If C is simplicial cochain algebra its realization $|C|$ is also an algebra, the product being defined by "Eilenberg–Zilber shuffles". That is, the product of $x \in C_{p_1}^{q_1}$ and $y \in C_{p_2}^{q_2}$ is defined as $\sum_\theta \operatorname{sign}(\theta) \cdot \theta_1^* x \cdot \theta_2^* y$ in $C_{p_1+p_2}^{q_1+q_2}$, where $\theta = (\theta_1, \theta_2): [p_1 + p_2] \to [p_1] \times [p_2]$ runs through all (p_1, p_2)-shuffles. If each of the algebras C_p is commutative then the product in $|C|$ is also commutative. (This is of course in contrast with the non-commutative Alexander–Whitney product which applies to cosimplicial algebras.)

If X is a smooth manifold its de Rham complex $\Omega(X)$ is a convenient commutative cochain complex which represents its homotopy type over the real numbers. The simplest commutative cochain complex for a general space seems to be the singular de Rham complex, for which a q-cochain is a compatible collection of elements of $\Omega^q(\Delta^r)$, one for each singular simplex $\Delta^r \to X$. This is discussed in [17]. Proposition (5.9) below enables us to pass rather freely between different kinds of cochains.

We shall suppose at any rate that our cochains are such that there are natural integration maps $\int_{\Delta^p}: C^{p+q}(\Delta^p \times X) \to C^q(X)$ for any space X satisfying $(-1)^p \, d \int_{\Delta_p} \alpha = \int_{\Delta^p} d\alpha - \int_{\partial\Delta^p} \alpha$ and vanishing on elements pulled back from $\Delta^{p-1} \times X$ by degeneracy maps $\Delta^p \to \Delta^{p-1}$. Then for any cosimplicial space Z the natural maps $\Delta^p \times |Z| \to Z_p$ induce homomorphisms $C^{p+q}(Z_p) \to C^q(|Z|)$ which fit together to define a cochain map $|C(Z)| \to C(|Z|)$. It is even simple to check that this is a homomorphism of algebras when the cochains are multiplicative, providing $\int_{\Delta^p}: C(\Delta^p \times X) \to C(X)$ is a homomorphism of $C(X)$-modules.

For our main theorem we shall consider only a special class of cosimplical spaces. Suppose that $\{P_\alpha\}_{\alpha \in A}$ is an inverse system of spaces, i.e. A is a partially ordered set and there are given compatible maps $P_\beta \to P_\alpha$ whenever $\alpha \leq \beta$. Then there is a cosimplicial space Z with $Z_p = \prod_{\alpha_0 \leq \cdots \leq \alpha_p} P_{\alpha_0}$. (Its realization is often called the "homotopical inverse limit" of the system.)

PROPOSITION (5.3) (Anderson[1]). *In the foregoing situation if A is finite and each P_α is n-connected, where n is the dimension of the nerve of A, then the natural map $|C(Z)| \to C(|Z|)$ is a cohomology equivalence.*

(Of course the *nerve* of A means the simplicial set whose p-simplexes are the chains $\alpha_0 \leq \cdots \leq \alpha_p$ in A.)

COROLLARY (5.4). *If U is a finite open covering of X with nerve Σ, and $\pi_i(Y) = 0$ for $i \leq \dim \Sigma$, then $|C(Y^\Sigma)| \to C(Y^{|\Sigma|})$ is a cohomology equivalence.*

The proof of (5.3) will be postponed to §7.

Without any more work we can now obtain a result for the space of sections of a bundle. First recall [14] that to an open covering $\{U_\alpha\}_{\alpha \in S}$ of a space X is associated a simplicial space X_Σ called its *thickened nerve*, defined by

$$(X_\Sigma)_p = \coprod_{\sigma_0 \leq \cdots \leq \sigma_p} U_{\sigma_0}.$$

There is a natural map $\pi: |X_\Sigma| \to X$, which is a homotopy equivalence if X is paracompact.

Now let $p: E \to X$ be a map, and let Γ_σ be the space of sections of p over U_σ. $\{\Gamma_\alpha\}$ is an inverse system of spaces, to which is associated the cosimplicial space Γ_Σ. We have the tautology.

PROPOSITION (5.5). *The realization of Γ_Σ is the space of sections of $\pi^* E$, the pull-back of E by $\pi: |X_\Sigma| \to X$.*

COROLLARY (5.6). *If $p: E \to X$ is a Hurewicz fibration and X is paracompact then $|\Gamma_\Sigma| \simeq \Gamma(E)$, the space of sections of E.*

If we now choose the covering U contractible the spaces Γ_σ from which Γ_Σ is formed are all homotopy equivalent to fibres of $E \to X$, and (5.3) and (5.6) give a cochain complex for $\Gamma(E)$ providing the covering is finite and of dimension less than the connectivity of the fibres. But because the spaces Γ_σ are infinite-dimensional it will be convenient to replace Γ_Σ by an equivalent but very much smaller cosimplical space, constructed as follows.

For any subset σ of S such that U_σ is non-empty let U^σ denote $\bigcup_{\alpha \in \sigma} U_\alpha$, the *star* of U_σ. This is contractible if U is a contractible covering. Let $E^\sigma = p^{-1}(U^\sigma)$. Then $\{E^\sigma\}$ is an inverse system of spaces. It defines a cosimplicial space E^Σ which is the model we are seeking. It is comparable in size with the cosimplicial space Y^Σ for a mapping space. The essential property of E^Σ follows from

PROPOSITION (5.7). *If $p: E \to X$ is a Hurewicz fibration on a paracompact base, and U is a contractible covering of X then there is a transformation of inverse systems $\{E^\sigma\} \to \{\Gamma_\sigma\}$ such that $E^\sigma \to \Gamma_\sigma$ is a homotopy equivalence for each σ.*

Proof. Consider the relation$\sim$ on X defined by $x \sim y$ if and only if x and y both belong to a single set U_α of the covering U. A *slicing function* (cf. [12]) for the bundle E is a family of homotopy equivalences $T_{xy}: E_x \to E_y$ defined when $x \sim y$ and depending continuously on (x, y). (Here E_x means $p^{-1}(x)$.) It is easy to see that any Hurewicz fibration on a paracompact base admits a slicing function with respect to any contractible covering. Using it one defines $E^\sigma \to \Gamma_\sigma$ by associating to $e \in E_x$ the section whose value at y is $T_{xy}e$. That is possible because if $x \in U^\sigma$ and $y \in U_\sigma$ then $x \sim y$. The naturality with respect to σ is immediate.

COROLLARY (5.8). *If X has a finite contractible covering of dimension n, and $E \to X$ is a Hurewicz fibration whose fibres are n-connected, then there is a cohomology equivalence $|C(E^\Sigma)| \to C(\Gamma(E))$.*

To deduce the corollary from the preceding proposition one needs the following lemma, which we will use repeatedly. It is the analogue of a familiar property [15, (A.1)] of the realization of simplicial spaces.

PROPOSITION (5.9). *If $C \to C'$ is a morphism of simplicial cochain complexes such that $C_p \to C'_p$ is a cohomology equivalence for each p then $|C| \to |C'|$ is a cohomology equivalence.*

Proof. The realization $|C|$ of a simplicial cochain complex is the direct limit of its "skeletons" $|C|_n$, where $|C|_n$ is the total complex of the truncated bicomplex $(C_0 \leftarrow \bar{C}_1 \leftarrow \cdots \leftarrow \bar{C}_n)$. Because cohomology commutes with direct limits it is enough to show $|C|_n \to |C'|_n$ is an equivalence for each n. But $|C|_n$ fits into a short exact sequence $0 \to |C|_{n-1} \to |C|_n \to \bar{C}_n \to 0$ of cochain complexes, so by induction and the five lemma it is enough to show $\bar{C}_n \to \bar{C}'_n$ is an equivalence. This follows from the lemma.

LEMMA (5.10). *For any simplicial cochain complex C there is an exact sequence of cochain complexes*

$$0 \leftarrow \bar{C}_n \leftarrow C_n \leftarrow \bigoplus_i C_{n-1}^{(i)} \leftarrow \bigoplus_{i<j} C_{n-2}^{(ij)} \leftarrow \cdots \leftarrow C_0^{(12\ldots n)} \leftarrow 0,$$

where $C_{n-k}^{(i_1 \ldots i_k)}$ is a copy of C_{n-k}.

§6. THE MAIN THEOREM

Let M be a Riemannian manifold which has a finite covering U by geodesically convex open sets. If M is either compact or the interior of a compact manifold with boundary it has such a covering, as any covering by sufficiently small geodesic disks will do [4].

Let P be the principal bundle of M, with group O_n, and let E be the associated bundle whose fibre is the O_n-manifold F constructed in §3.

PROPOSITION (6.1). *The Gelfand–Fuchs algebra $A(M)$ has the homotopy type of the space of sections of E (over the complex numbers).*

This means of course that $A(M)$ is linked to the complex cochain algebra of $\Gamma(E)$ by a sequence of homomorphisms of cochain algebras (unfortunately not all going in the same direction) each of which is a cohomology equivalence.

In §5 we showed that a suitable cochain algebra for the space of sections is given by the realization of the simplicial algebra $\Omega(E^\Sigma)$, where E^Σ is the cosimplical space associated to the contractible covering U. Proposition (5.8) applies because the fibre F is $2n$-connected when M is n-dimensional. (Homotopically F is the fibre of the inclusion of the $2n$-skeleton of BU_n in BU_n.) One can use the differential forms on E^Σ as cochains by (5.9), for each term of E^Σ is a finite dimensional manifold.

The contractible covering U can also be used to decompose the Gelfand–Fuchs cochains $A(M)$. Because A is a covariant functor for immersions one can apply it to the thickened nerve M_Σ described in §5 to obtain a simplicial cochain algebra $A(M_\Sigma)$ with an augmentation to $A(M)$.

PROPOSITION (6.2). *The augmentation induces a cohomology equivalence*

$$|A(M_\Sigma)| \to A(M).$$

The proof of (6.2) will be given in §8. In fact a familiar argument with a partition of unity shows that the simplicial algebra $A(M_\Sigma)$ is a resolution of $A(M)$, and (6.2) would follow immediately if there were not considerable difficulties with convergence.

Granting (6.2) our task is to relate the simplicial algebras $A(M_\Sigma)$ and $\Omega(E^\Sigma)$. For each point p of the principal bundle P of M, i.e. each frame in the tangent bundle, there is an exponential map $\exp_p : \mathbf{R}^n \to M$ because U_α is geodesically convex this gives us an identification of U_α with an open set of $\mathbf{R}^n$ for each point of P_α, the part of the principal bundle over U_α. Thus P_α can be thought of as a family of embeddings of U_α in $\mathbf{R}^n$, and $P^\sigma = \bigcup_{\alpha \in \sigma} P_\alpha$ as a family of embeddings of U_σ in $\mathbf{R}^n$. The map $P^\sigma \times U_\sigma \to \mathbf{R}^n$ is equivariant with respect to the action of O_n on P^σ and $\mathbf{R}^n$. From §4 it induces a cochain homomorphism $A(U_\sigma) \to \Omega(P^\sigma; A(\mathbf{R}^n))_{\text{basic}}$.

On the other hand $E^\sigma = (P^\sigma \times F)/O_n$, so $\Omega(E^\sigma)$ can be identified with $\Omega(P^\sigma \times F)_{\text{basic}}$, i.e. with $\Omega(P^\sigma; \Omega(F))_{\text{basic}}$. Recalling from §3 the O_n-homomorphisms $A(\mathbf{R}^n) \leftarrow A \to \Omega(F)$, we have a chain

$$A(U_\sigma) \to \Omega(P^\sigma; A(\mathbf{R}^n))_{\text{basic}} \leftarrow \Omega(P^\sigma; A)_{\text{basic}} \to \Omega(P^\sigma; \Omega(F))_{\text{basic}} = \Omega(E^\sigma),$$

which is clearly natural with respect to σ.

All of these morphisms are cohomology equivalences. For $P^\sigma = O_n \times U^\sigma$ as O_n-space (non-canonically), so the chain can be rewritten

$$A(U_\sigma) \to \Omega(U^\sigma; A(\mathbf{R}^n)) \leftarrow \Omega(U^\sigma; A) \to \Omega(U^\sigma; \Omega(F)) = \Omega(E^\sigma).$$

Here the three terms of the form $\Omega(U^\sigma; B)$ are double complexes which (by de Rham's theorem) can be thought of as resolutions of B, so that $B \to \Omega(U^\sigma; B)$ is a cohomology equivalence, and the chain is equivalent to

$$A(U_\sigma) \to A(\mathbf{R}^n) \leftarrow A \to \Omega(F),$$

in which all the maps are equivalences—the first by (2.1).

Natural transformations linking $A(U_\sigma)$ and $\Omega(E^\sigma)$ are not exactly what is needed to link $A(M_\Sigma)$ and $\Omega(E^\Sigma)$. We need rather a relation between $A(\coprod_\sigma U_\sigma)$ and $\Omega(\prod_\sigma E^\sigma)$, where σ runs through the p-simplexes of the nerve of U. But $\prod_\sigma P^\sigma$ is a space of immersions of $\coprod_\sigma U_\sigma$ in $\coprod^k \mathbf{R}^n$ (where k is the the number of p-simplexes), and $\prod_\sigma E^\sigma = \{(\prod_\sigma P^\sigma) \times F^k\}/G$, where $G = (O_n)^k$. The foregoing argument applies without change to this multiplied situation, providing one knows that the maps $A(\coprod^k \mathbf{R}^n) \leftarrow A^{\otimes k} \to \Omega(F^k)$ are cohomology equivalences. For the right-hand map this follows from the usual Künneth theorem, as F has finite dimensional cohomology. For the other, we saw in §2 that there is an equivalence $A_0(\coprod^k \mathbf{R}^n) \to A(\coprod^k \mathbf{R}^n)$. But $A_0(\coprod^k \mathbf{R}^n) = A_0(\mathbf{R}^n)^{\otimes k}$, which is equivalent to $A^{\otimes k}$ by the Künneth theorem.

Thus we have found equivalences linking $A(M_\Sigma)$ and $\Omega(E^\Sigma)$, and so completed the proof of (6.1).

§7. PROOF OF (5.3)

We have seen that for any cosimplicial space X there is a natural map $|H^*(X)| \to H^*(|X|)$. We shall say that X *converges* when it is an isomorphism.

The starting point of our proof is the "Eilenberg-Moore spectral sequence" [6]. This amount to the assertion that the cosimplicial space Π associated to the diagram $(X \to Z \leftarrow Y)$, i.e.

$$\Pi = \{X \times Y \rightrightarrows X \times Z \times Y \rightrightarrows X \times Z \times Z \times Y \rightrightarrows \cdots\},$$

converges providing the space Z is connected and simply connected. The realization of Π is the homotopical fibre product of X and Y over Z, which we shall denote by $X\pi_Z Y$. One should observe that the hypothesis on Z implies that the total complex $|H^*(\Pi)|$ involves only finitely many $H^{p+r}(\Pi_p)$ in each total dimension r, for $H^{p+r}(\Pi_p)$ will consist of degenerate elements when $p > r$.

If now $X \to Z \leftarrow Y$ is a diagram of cosimplicial spaces instead of spaces then the foregoing Π will be a bi-cosimplicial space. It has a total realization $|\Pi|$ which can be formed either by realizing the columns first, giving $|X|\pi_{|Z|}|Y|$, or else by realizing the rows first, giving $|X\pi_Z Y|$. That is to say, the realization of cosimplicial spaces commutes with homotopical fibre products.

LEMMA (7.1). *If the cosimplicial spaces X, Y, Z converge then $X\pi_Z Y$ converges, providing*
(i) *$|Z|$ and each Z_p is connected and simply connected, and*
(ii) *$|X|$, $|Y|$, $|Z|$ and each X_p, Y_p, Z_p are of finite type,*
i.e. *their cohomology is finitely generated in each dimension.*

Proof. One realizes in two ways the bi-simplicial graded abelian group $\{C_{pq}\}$ with $C_{pq} = H^*(X_q \times Z_q^{p-1} \times Y_q)$, and applies the Eilenberg–Moore theorem to each row. One needs to know that the product of convergent cosimplicial spaces is convergent. That is true because $|H^*(X \times Y)| \cong |H^*(X) \otimes H^*(Y)| \cong |H^*(X)| \otimes |H^*(Y)|$ by the Künneth theorem and the Eilenberg–Zilber theorem respectively. Using the Künneth theorem makes the hypotheses of finite type necessary, but they could be avoided by considering homology instead of cohomology. (One assumes the coefficients are a field to apply the Künneth theorem, but can then immediately remove the assumption.)

Now suppose that Σ is a simplicial set and $\sigma \mapsto Q_\sigma$ is a covariant functor from the simplexes of Σ to spaces. There is a cosimplicial space

$$Q^\Sigma = \left\{ \prod_{\sigma \in \Sigma_0} Q_\sigma \rightrightarrows \prod_{\sigma \in \Sigma_1} Q_\sigma \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right\}.$$

The situation of (5.3) is the particular case where Σ is the nerve of the partially ordered set A, and the functor is $[\alpha_0 \leq \cdots \leq \alpha_p] \mapsto P_{\alpha_0}$. Using (7.1) we propose to prove Q^Σ is convergent by induction on the number of non-degenerate simplexes in Σ.

If Σ is the union of two simplicial sets Σ^1 and Σ^2 whose intersection is Σ^0 then the cosimplicial space Q^Σ is the fibre-product of Q^{Σ^1} and Q^{Σ^2} over Q^{Σ^0}. We should like to apply (7.1) to show that Q^Σ converges if the Q^{Σ^i} do. The fibre-product of Q^{Σ^1} and Q^{Σ^2} over Q^{Σ^0} maps to their homotopical fibre-product by a map which is obviously a homotopy equivalence at each level, so (7.1) can be applied providing the fibre-product and the homotopical fibre-product of $|Q^{\Sigma^1}|$ and $|Q^{\Sigma^2}|$ over $|Q^{\Sigma^0}|$ are equivalent. For this it suffices for $|Q^{\Sigma^1}| \to |Q^{\Sigma^0}|$ to be a fibration, which is easily seen to be the case. (Strictly speaking, one ought to say that a cosimplicial space of the type of Q^Σ is *good* (cf. [15] App. A) in the sense that its degeneracy maps are fibrations, and a map of good cosimplicial spaces induces an equivalence of their realizations of it is an equivalence at each level.)

The connectivity hypotheses in (7.1) will hold for Q^{Σ^i} providing each Q_σ is n-connected and $n > \dim \Sigma^i$. If so then the finiteness hypotheses will hold providing each Q_σ is of finite type, and Σ^i is finite, i.e. has only finitely many non-degenerate simplexes.

Thus for a finite simplicial set Σ to prove Q^Σ converges it is enough, by induction, to show that Q^Ω converges, where Ω runs through the irreducible sub-simplicial-sets of Σ, which consist of the faces of a single non-degenerate simplex. In proving (5.3) the Ω which arise are the nerves of totally ordered sets $A = \{\alpha_1 \leq \alpha_1 \leq \cdots \leq \alpha_p\}$. Then $|Q^\Omega| \simeq Q_{\alpha_0}$, and on the other hand $H^*(Q^\Omega)$ is a resolution of $H^*(Q_{\alpha_0})$, for the simplicial homotopy which contracts Ω induces a chain homotopy contracting $H^*(Q^\Omega)$ to $H^*(Q_{\alpha_0})$. The completes the proof of (5.3).

§8. PROOF OF (6.2)

If one has a double complex with augmentation

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
A^2 & \leftarrow & C_0^2 & \leftarrow & C_1^2 & \leftarrow & C_2^2 & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
A^1 & \leftarrow & C_0^1 & \leftarrow & C_1^1 & \leftarrow & C_2^1 & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
A^0 & \leftarrow & C_0^0 & \leftarrow & C_1^0 & \leftarrow & C_2^0 & \leftarrow & \cdots
\end{array}$$

such that each row is an exact sequence, then A has the same cohomology as the "infinite total complex" whose cochains in dimension r are the product $\prod_{q\geq 0} C_r^{q+r}$. This is not the total complex in the sense we are using, for we take the direct sum rather than the product. Apart from this difficulty Proposition (6.2), which concerns the augmented double complex with $A^p = A^p(M)$ and $C_q^p = A^p(M_{\Sigma,q})$, where $M_{\Sigma,q}$ is the qth space of the simplical space M_Σ, would be almost trivial, for the rows form exact sequences by a standard argument using a partition of unity.

To deal with this "convergence" problem we shall use the Gelfand–Fuchs filtration on the cochain algebra $A(M)$. An element of $A^p(M)$ is a distribution on the product of p copies of M. We say it has filtration $\leq k$ if its support is contained in M_k^p, the subset of points $(x_1, \ldots, x_p)$ of M^p such that at most k of the x_i are distinct. If $A_k^p(M)$ denotes the cochains of filtration $\leq k$ then

$$0 = A_0^p(M) \subset A_1^p(M) \subset \cdots \subset A_p^p(M) = A^p(M).$$

Suppose that for some k we form an augmented double complex by applying the functor A_k instead of A to the augmented simplicial space $M_\Sigma \to M$. We shall prove

(a) each row is degenerate above dimension dk, where $d = \dim(\Sigma)$, and

(b) each row is an exact sequence.

The statements (a) and (b) imply that $|A_k(M_\Sigma)| \to A_k(M)$ is a cohomology equivalence, because the normalized double complex associated to $A_k(M_\Sigma)$ has then only $kd + 1$ non-zero columns. But taking the union for all k we deduce that $|A(M_\Sigma)| \to A(M)$ is an equivalence, proving (6.2).

Proof of (*a*) (cf. [5, (4.18)]. Let F be a covariant functor from smooth manifolds and immersions to abelian groups. We shall say that F is of degree $\leq k$ if for any manifolds $X_1, \ldots, X_m$ (with $m \geq k$) the natural map

$$\bigoplus_{i_1 < \cdots < i_k} F(X_{i_1} \amalg \cdots \amalg X_{i_k}) \to F(\coprod_{1 \leq i \leq m} X_i)$$

is surjective. The functor A_k^p is obviously of degree k.

If $U = \{U_\alpha\}_{\alpha \in S}$ is an open covering of M the simplicial space M_Σ is defined by $M_{\Sigma,q} = \coprod_{\sigma_0 \leq \cdots \leq \sigma_1} U_{\sigma_0}$. The dimension of Σ is the largest d such that there is a chain $\sigma_0 < \sigma_1 < \cdots < \sigma_d$ of subsets of S with U_{σ_0} non-empty, i.e. such that M_Σ is degenerate above dimension d.

LEMMA (8.1). *If F is of degree k the simplicial vector space $F(M_\Sigma)$ is degenerate above dimension kd, where $d = \dim(\Sigma)$.*

Proof. Suppose $q > kd$. In dimension $qF(M_\Sigma)$ is $F(\coprod_c U_c)$, where c runs through the chains $c = (\sigma_0 \leq \cdots \leq \sigma_q)$ of Σ, and U_c denotes U_{σ_0}. So each element of it is a sum of elements coming from $F(U_{c_1} \amalg \ldots \amalg U_{c_k})$ for some chains $c_1, \ldots, c_k$. Bit each chain c_i has no more than d non-zero steps, so there must be at least one j such that the jth step of c_i is zero for each i. Then all of $F(U_{c_i} \amalg \cdots \amalg U_{c_k})$ is in the image of the jth degeneracy operation. This proves the lemma.

Proof of (*b*). We have to construct a filtration-preserving contracting homotopy of the simplicial vector space $A^p(M_\Sigma)$.

The simplicial space M_Σ comes from a finite open covering $\{U_\alpha\}_{\alpha \in S}$ of M. By choosing a shrinkage $\{U'_\alpha\}$ of $\{U_\alpha\}$, i.e., $\bar{U}'_\alpha \subset U_\alpha$, one obtains a sub-simplicial space M'_Σ of M_Σ. The complex $A^p(M_\Sigma)$ consists of distributions with compact support on the spaces of $(M_\Sigma)^p$, so it is the union of the $A^p(M'_\Sigma)$ for all shrinkages $\{U'_\alpha\}$ of $\{U_\alpha\}$. It will therefore suffice for us to give a filtration-preserving null-homotopy of the inclusion $A^p(M'_\Sigma) \to A^p(M_\Sigma)$.

Now for any manifold M $A^p(M)$ is the invariant subspace under an action of the pth symmetric group $\mathcal{S}_p$ in a larger space $B^p(M)$ obtained by omitting the alternating condition on cochains. Taking $\mathcal{S}_p$-invariants is an exact functor, so it will suffice for us to give a null-homotopy of $B^p(M'_\Sigma) \to B^p(M_\Sigma)$.

If $\pi = (\sigma_1, \ldots, \sigma_p)$ is a p-tuple of subsets of S let U_π denote $U_{\sigma_1} \times \cdots \times U_{\sigma_p}$. The set of such p-tuples π is partially ordered set Π such that $\pi \leq \pi' \Rightarrow U_\pi \subset U_{\pi'}$. The qth term of the simplicial space $(M_\Sigma)^p$ is

$$\coprod_{\pi_0 \leq \cdots \leq \pi_q} U_{\pi_0},$$

and the action of $\mathcal{S}_p$ on $(M_\Sigma)^p$ comes from its action on Π. We define U'_π similarly, using the shrinkage $\{U'_\alpha\}$ of $\{U_\alpha\}$.

For each $\pi \in \Pi$ define $W_\pi = U_\pi - \bigcup_{p \geq \pi} \bar{U}_p$. Then $\{W_\pi\}$ is an open covering of M^p, for if $x \in M^p$ then $x \in W_\pi$, where $\pi = \inf\{p : x \in U_p\}$. Let $\{\varphi_\pi\}$ be a partition of unity subordinate to $\{W_\pi\}$. The vector space $B^p(M'_{\Sigma,q})$ is spanned by elements which can be denoted $[\pi_0\pi_1 \cdots \pi_q; f]$, where $\pi_0 \leq \cdots \leq \pi_q$ in Π, and f is a distribution with compact support in U'_{π_0}. Then

$$\partial : B^p(M'_{\Sigma,q}) \to B^p(M'_{\Sigma,q-1})$$

is given by

$$\partial[\pi_0 \cdots \pi_q; f] = \Sigma(-1)^i[\pi_0 \cdots \hat{\pi}_i \cdots \pi_q; f].$$

We define the desired chain-homotopy

$$h : B^p(M'_{\Sigma,q}) \to B^p(M'_{\Sigma,q+1})$$

by

$$h[\pi_0 \cdots \pi_q; f] = \sum_\pi [\pi\pi_0 \cdots \pi_q; \varphi_\pi f].$$

This makes sense because if $\pi_0 \not\geq \pi$ the sets W_π and U'_{π_0} are disjoint, and $\varphi_\pi f = 0$. It is trivial that $\partial h + h\partial$ is the inclusion; so h is a null-homotopy. The crucial point is to show that if $[\pi_0 \cdots \pi_q; f]$ has filtration k then so does $[\pi\pi_0 \cdots \pi_q; \varphi_\pi f]$, and for this to hold one must choose the partition $\{\varphi_\pi\}$ appropriately. The condition needed is that for every $x \in \operatorname{supp}(\varphi_\pi) \subset M^p$, and every $g \in \mathcal{S}_p$ such that $gx = x$, one has $g\pi = \pi$. That is, one needs

$$\operatorname{supp}(\varphi_\pi) \subset W'_\pi = W_\pi - \bigcup_{g\pi \neq \pi} (M^p)^g.$$

But $\{W'_\pi\}$ is an open covering of M^p, for if $x \in M^p$ and $\pi = \inf\{p : x \in U_p\}$ then $x \in W'_\pi$. So $\{\varphi_\pi\}$ can be chosen subordinate to $\{W'_\pi\}$, and the proof is complete.

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